ECON 7010 - Macroeconomics I

Fall 2015

Notes for Lecture #2

Today- Cake eating problem:

- $T = 2$
  - Endowment in period 2
  - Return on storage
- $T > 2$

The problem:

- $V_2(w_1) = \max_{w_1 \geq s \geq 0} u(w_1 - s) + \beta u(s)$
  - recall that V_2 is called the value function
  - recall that $w_3 = 0$ b/c eat all cake in two periods
- $\Rightarrow u'(w_1 - s) = \beta u'(s)$ (i.e., marginal utilities, discounted, are equal) (EXPLAIN to class that just need this one FOC because we included transition equation into model and know it'll be an interior solution due to our assumptions on $u(\cdot)$)
- $\Rightarrow s(w_1)$ policy function (or decision rule) $\rightarrow$ so $\forall$ w we have solved the problem
  - recall that the policy function reflects preferences: $u(c)$, β
  - goal of empirical research is to learn about these preferences
  - empirical research will take for granted that individuals optimize to help identify preferences
  - Preferences and technologies form decisions rules, which are derived from the data (i.e., we observe decisions through economic outcomes)

Examples:

1. $\beta = 1$, any concave utility function
   - $V_2(w_1) = u(w_1 - s) + u(s)$
   - Euler equation says that $u'(c_1) = \beta u'(c_2)$
   - $\Rightarrow c_1 = c_2 = \frac{w}{2}$
   - $V_2 = 2u\left(\frac{w}{2}\right)$
   - Why?
     - Euler equation says that $u'(c_1) = \beta u'(c_2)$
     - Because $u''(c) < 0$, agent is risk averse and would like to smooth consumption
     - This is an extreme example of consumption smoothing since $\beta = 1$, there is no discounting of future consumption
     - And since there is no uncertainty, there is no precautionary savings
     - Thus have $u'(c_1) = u'(c_2) \Rightarrow c_1 = c_2$
   - Draw two axes with c_1 and c_2 and 45 degree line. Show preferences tangent right at 45 degree line, along budget constraint which goes from w_1 on one axis to the other.
2. $\beta < 1$, $u(c) = ln(c)$

- $V_2(w_1) = ln(w_1 - s) + \beta ln(s)$
- Euler equation $\Rightarrow u'(c_1) = \beta u'(c_2) < u'(c_2)$
  - $u''(\cdot) < 0 \Rightarrow c_1 > c_2$
  - Euler: $\frac{1}{c_1} = \frac{\beta}{c_2}$
  - or, $\frac{1}{w_1 - s} = \frac{\beta}{s} \Rightarrow s = \frac{\beta w_1}{1+\beta} = c_2$
  - $\Rightarrow c_1 = \frac{w_1}{1+\beta}$
  - $V_2(w_1) = ln\left(\frac{w_1}{1+\beta}\right) + \beta ln\left(\frac{\beta w_1}{1+\beta}\right)$
  - or $V_2(w_1) = ln\left(\frac{1}{1+\beta}\right) + \beta ln\left(\frac{\beta}{1+\beta}\right) + (1+\beta)ln(w_1)$ (get this by using properties of logs)
- Draw two axes with c_1 and c_2 and 45 degree line. Show preferences tangent below 45 degree line, along budget constraint which goes from w_1 on one axis to the other.

Implicit Functions: (a sidebar)

- Suppose you have a function like: $G(x, \alpha) = c$, where c is a constant (e.g., 0)
- For each α, there is a value of x that makes $G(x, \alpha) = c$, call this value of x, x^*
  - Therefore, $G(x, \alpha) = c$ implicitly defines x^* for each α
  - x^* is an implicit function of α
  - Thus we can write $G(x^*(\alpha), \alpha) = c$
- Differentiating an implicit function:
  - b/c $G(x^*(\alpha), \alpha) = c$, we know
  - $\frac{dG(x^*(\alpha),\alpha)}{d\alpha} = 0$
  - Using the chain rule, we get:
  - $\frac{dG(x^*(\alpha),\alpha)}{d\alpha} = \underbrace{\frac{\partial G(x^*(\alpha), \alpha)}{\partial x^*(\alpha)}}_{G_1(x^*(\alpha),\alpha)} \frac{dx^*(\alpha)}{d\alpha} + \underbrace{\frac{\partial G(x^*(\alpha), \alpha)}{\partial \alpha}}_{G_2(x^*(\alpha),\alpha)} = 0$
  - $\Rightarrow \frac{dx^*(\alpha)}{\alpha} = \frac{-G_2(x^*(\alpha),\alpha)}{G_1(x^*(\alpha),\alpha)}$
  - Now we have an equation for the derivative of the implicit function
- The Implicit Function Theorem (For a function of two variables):
  - Let $G(x, \alpha)$: $\mathbb{R}^2 \rightarrow \mathbb{R}$ be a continuously differentiable function. Consider a pair (x_0, α_0) satisfying $G(x_0, \alpha_0) = c$. If $G_1(x_0, \alpha_0) \neq 0$, then there exists a continuously differentiable function $x^*(\alpha)$ defined on the interval I about point α_0 such that:
    1. $G(x^*(\alpha), \alpha) = c, \quad \forall c \in I$
    2. $x^*(\alpha_0) = x_0$
    3. $\frac{dx^*}{d\alpha}|_{\alpha_0} = \frac{-G_2(x_0,\alpha_0)}{G_1(x_0,\alpha_0)}$
  - Example:
    * Let $G(x^*(\alpha), \alpha) = x - 2\alpha = 3$
    * Want $\frac{dx^*}{d\alpha}$
    * IFT $\Rightarrow \frac{dx^*}{d\alpha}|_{\alpha_0} = \frac{-G_2(x_0,\alpha_0)}{G_1(x_0,\alpha_0)}$
    * $G_1 = 1, \forall x, \alpha$
    * $G_2 = -2, \forall x, \alpha$

* $\Rightarrow \frac{dx^*}{d\alpha} = -\frac{-2}{1} = 2$
* Confirm by solving for x as a function of α:
* $x - 2\alpha = 3$
* $\Rightarrow x = 3 + 2\alpha$
* $\Rightarrow \frac{dx^*}{d\alpha} = 2$

- Implicit functions in optimization problems:
  - Our FOCs define implicit functions
  - That is, they implicitly define the values of the endogenous variables for given values of exogenous variables
  - e.g. Euler equation
    * $u'(w_1 - s) = \beta u'(s)$
    * $\Rightarrow u'(w_1 - s) - \beta u'(s) = 0 = G(s^*(w_1), w_1)$
    * So, if we want to find the change in the endogenous variable s for a change in the exogenous variable w_1, we can use the IFT:
    * $\frac{ds}{dw_1} = \frac{-G_2}{G_1} = \frac{u''(w_1-s)}{u''(w_1-s)+\beta u''(s)}$

Comparative statics (see how endogenous variable changes for changes in exogenous variable):

- FOC: $u'(w_1 - s) - \beta u'(s) = 0$
- Use IFT to find comparative static as shown above
  - $\frac{ds}{dw_1} = \frac{u''(c_1)}{u''(c_1)+\beta u''(c_2)} > 0$ (because both numerator and denominator are negative due to assumption of convex utility function)
  - but, b/c FOC says $\beta = \frac{u'(c_1)}{u'(c_2)}$, we can rewrite as:
  - $\frac{ds}{dw_1} = \frac{1}{1+\beta\frac{u''(c_2)}{u''(c_1)}} = \frac{1}{1+\frac{u''(c_2)/u'(c_2)}{u''(c_1)/u'(c_1)}} = \frac{1}{1+ratio\ of\ risk\ aversion} = \frac{1}{1+\frac{A(c_2)}{A(c_1)}} = \frac{A(c_1)}{A(c_1)+A(c_2)} < 1$
    * Where: $A(x) \equiv$ The coefficient of absolute risk aversion
      · $A(x) \equiv \frac{-u''(x)}{u'(x)}$
    * that $\frac{ds}{dw_1} > 0$, means that c_2 increases with increases in cake endowment
    * that $\frac{ds}{dw_1} < 1$ means that c_1 increases with increases in cake endowment (i.e, agent doesn't save all of increase in cake for second period)
    * $\Rightarrow$ savings function is increasing and always has a slope of less than one in this case
    * $0 < \frac{ds}{dw_1} < 1$

Extension #1: endowment of cake in period 2 (y_2)

- Preferences: $u(c_1) + \beta u(c_2)$
- Endowment: w_1 when born, y_2 in period 2
- Constraints:
  - $w_2 = w_1 - c_1$
  - $c_2 = y_2 + w_2$
  - or $c_1 + c_2 = w_1 + y_2$ (combining first two constraints)

  - Note that we are imposing $w_3 = 0$, this is b/c $w_3 > 0$ is suboptimal and $w_3 < 0$ is not allowed
  - Note that $c_1 > w_1$ is ok; borrowing is allowed
  - Can rewrite first two constraints in terms of savings (where $s = w_2$):
    * $s = w_1 - c_1$
    * $c_2 = y_2 + s$

- Problem: $\max_s u(w_1 - s) + \beta u(y_2 + s)$
- FOC: $u'(w_1 - s) = \beta u'(y_2 + s)$
  - $s(w_1, y_2)$
  - $c_1(w_1, y_2) = c_1(w_1 + y_2)$
  - $c_2(w_1, y_2) = c_1(w_1 + y_2)$
    * Consumption just depends upon the sum of w_1 and y_2, not the timing (since can borrow/lend freely) - all that matters is the lifetime endowment
    * Savings does depend upon the timing of the endowments - that's the whole point of savings, to change the timing of consumption
- Draw two axes with c_1 and c_2 and 45 degree line. Show preferences tangent below 45 degree line, along budget constraint which goes from $w_1 + y_2$ on one axis to the other. Label this the lifetime endowment.

A couple of examples of this extension:

1. Tax policy (Ricardian equivalence)
   - It doesn't matter if finance gov't spending with debt or current taxes
   - debt is future taxes
   - consumption will remain the same (lifetime endowments unchanged, just change in when taxes come), but savings changes, depending on if tax now and give back later or have no tax
2. Borrowing restrictions ($s \geq 0$)
   - With restrictions to borrowing, then consumption depends upon timing of income $\rightarrow$ b/c can consumer more in c_1.
   - Draw two axes with c_1 and c_2 and 45 degree line. Show preferences tangent below 45 degree line, along budget constraint which goes from $w_1 + y_2$ on one axis to the other. Label this the lifetime endowment. But draw endowment of w_1 and y_2 to the left of the 45 degree line. Say that borrowing constraint means that can't get to where want to be.

<u>Extension #2</u>: Return on storage

- $w_1 > 0$, $y_2 = 0$
- $\rho \equiv$ return to storage (more like a real than nominal rate of return) $\rightarrow$ it's a statement about storage technology
- $\Rightarrow$ transition equation $w_2 = \underbrace{(w_1 - c_1)}_{s} \rho$
- Problem: $\max_{w_1 \geq s \geq 0} u(w_1 - s) + \beta u(\rho s)$
  - FOC: $u' \underbrace{(w_1 - s)}_{c_1} = \beta \rho u' \underbrace{(\rho s)}_{c_2}$

- $\Rightarrow s(\rho, w_1)$ is a policy function
- $V_2(\rho, w_1)$ is the value function
- Draw axes with c_1 and c_2, budget constraint going from ρw_1 on the c_2 axis to w_1 on the c_1 axis, and a 45 degree line
  - if $\beta\rho = 1$, then indifference curve hits at 45 degree line
  - $\beta\rho = 1 \Rightarrow c_1 = c_2$
  - $\beta\rho < 1 \Rightarrow c_1 > c_2$
  - $\beta\rho > 1 \Rightarrow c_1 < c_2$

- Comparative Statics: w.r.t. (ρ, w_1)
  - Euler: $u'(w_1 - s) - \beta\rho u'(\rho s) = 0 \implies G(s(\rho), \rho)$
  - IFT $\Rightarrow \frac{ds}{d\rho} = \frac{-G_2}{G_1} = \frac{\beta u'(\rho s) + \beta\rho s u''(\rho s)}{-u''(w_1 - s) - \beta\rho^2 u''(\rho s)}$
  - $\frac{ds}{d\rho} = \beta \frac{[\overbrace{u'(s\rho)}^{substitution} + \overbrace{s\rho u''(s\rho)}^{income}]}{z} = \beta \frac{u'(s\rho)[1 - R(s\rho)]}{z}$, where $R(x) = \frac{-xu''(x)}{u'(x)}$
    - $z = -u''(c_1) - \beta\rho^2 u''(s\rho)$
    - $R(x)$ is the coefficient of relative risk aversion (relative to wealth/consumption)
    - more curvature in $u(\cdot)$ increases the measure of relative risk aversion
  - if $R(s\rho) < 1$, then $\frac{ds}{d\rho} > 0$
  - if $R(s\rho) > 1$, then $\frac{ds}{d\rho} < 0$
    - depends if income or substitution effect dominates
    - subs $= u'(s\rho) \rightarrow$ how utility changes with change in ρ
    - income $= u''(s\rho) * \underbrace{s\rho}_{\text{savings} \times \text{storage rate}} \rightarrow$ how marginal utility changes for a change in ρ
  - Draw graph with s and ρ on axes and a backwards bending curve (i.e. s rises with ρ to some point, then declines as ρ gets sufficiently large
  - If substitution effect is bigger, then a higher ρ = save more $\Rightarrow c_1$ is less
    - draw a graph with two budget constraints showing changes in ρ and have to indifference curves showing c_1 declining as ρ increases.
  - For comparative static w.r.t. change in w_1
    - Euler + IFT $\Rightarrow \frac{ds}{dw_1} = \frac{u''(w_1 - s)}{u''(w_1 - s) + \beta\rho^2 u''(\rho s)} = \frac{<0}{<0} > 0$
    - $\Rightarrow$ as endowment $\uparrow$, save more
  - Likewise, can do comparative statics w/ value function (an endogenous variable):
  - $V_2(w_1, \rho) = u(w_1 - s) + \beta u(\rho s) = u(w_1 - s(w_1, \rho)) + \beta u(\rho s(w_1, \rho))$
  - $\frac{dV_2}{d\rho} = -u'(c_1)\frac{ds}{d\rho} + \rho\beta u'(c_2)\frac{ds}{d\rho} + \beta s u'(c_2)$
  - $\Rightarrow \frac{dV_2}{d\rho} = \frac{ds}{d\rho}[\underbrace{-u'(c_1 + \rho\beta u'(c_2)}_{=0 \text{ by FOC}}] + \beta s u'(c_2)$
  - The above is an example of an envelope condition - change in ρ has no effect on s b/c V is at a maximum (and derivative at max=0)
  - only direct effect of change in ρ affects V_2
  - so just have: $\frac{dV_2}{d\rho} = \frac{\beta u(s(\cdot)\rho)}{d\rho}$
  - $= s(\cdot)\beta u'(s(\cdot)\rho) > 0$
  - Could do the same with $\frac{dV_2}{dw_1}$ and find value function increases as endowment increases...

- Specific example:
  - if $u(c) = ln(c)$
    - FOC: $\frac{1}{c_1} = \frac{\beta\rho}{c_2} = \frac{\beta\rho}{\rho s} = \frac{\beta}{s}$
    - $\Rightarrow s = \frac{\beta w_1}{1+\beta}$
    - $\Rightarrow \frac{ds}{d\rho} = 0$
    - That is, if you have a log utility function, the income and substitution effects will cancel out
      - subs $= u'(s\rho)$
      - income $= s\rho u''(s\rho)$
      - w/ log:
      - subs $= \frac{1}{s\rho}$
      - income $= \frac{-1}{s\rho}$
      - $\Rightarrow$ subs+income $= 0$
    - $\Rightarrow \frac{ds}{d\rho} = 0 \Rightarrow \varepsilon_{s,\rho} = 0 \rightarrow$ elasticity of savings w.r.t. savings technology is 0
    - finding a policy function from Euler equation and resource constraint:
      - Resource constraint: $c_1 = w_1 - s$
      - Euler: $\frac{1}{c_1} = \frac{\beta\rho}{c_2} = \frac{\beta\rho}{\rho s} = \frac{\beta}{s}$
      - $\Rightarrow \frac{1}{w_1 - s} = \frac{\beta}{s} \Rightarrow s = \left(\frac{\beta}{1+\beta}\right) w_1 = s(w_1)$ policy function
      - this implies $c_1 = \frac{w_1}{\beta+1}$, which is also a policy function

<u>Extension #3</u>: The T Period problem

- $\max_{c_1,...,c_T} \sum_{t=1}^{T} \beta^t u(c_t)$
- in general we have $G(c_1, ...c_T)$ (G could allow for different discounting, utility functions in different periods), but here we assume geometric discounting and additively separable utility
- $u' > 0, u'' < 0, u'(0) = \infty$
- s.t. $w_1 = \sum_{t=1}^{T} c_t$ (i.e., no flows in future periods; $\rho = 1$)
- This problem generates $T-1$ FOCS's: (w/ $T-1$ Lagrange multipliers - for $T-1$ transition equations)
  - $u'(c_t) = \beta u'(c_{t+1}), t = 1, ..., T-1 \Rightarrow$ also know $w_{T+1} = 0$ (i.e., no left over cake when optimize)
    - $\Rightarrow$ can't be better off by switching around when consume
  - $\Rightarrow u'(c_1) = \beta u'(c_2) = \beta^2 u'(c_3)...$
- Policy Function: (generated by a given β and $u(\cdot)$)
  - θ is parameter describing $u(\cdot)$
  - β = discount factor
  - $(\theta, \beta) \Rightarrow$ policy function and value function
    - $c_t(w_1), t = 1, ..., T \rightarrow$ optimal solution to the problem
- Value function
  - $V_T(w_1) = \sum_{t=1}^{T} \beta^t u(c_t(w_1)), \forall w_1$
    - put in the optimal c_t's and get the total value of the problem $\rightarrow$ the max utility
    - e.g., $u(c) = \frac{c^{1-\gamma}}{1-\gamma} \Rightarrow R(c) = \gamma$
  - Suppose you have solved the T-period problem and now you must solve the T+1 period problem, 2 options:

1. Solve $\max_{c_1,\ldots,c_T} \sum_{t=0}^{T} \beta^t u(c_t)$
   * s.t. $w_0 = \sum_{t=0}^{T} c_t$
2. solve $\max_{c_0} u(c_0) + V_T(w_1) \to V_T$ is the value function for the T period problem
   * $w_1 = w_0 - c_0 \Rightarrow$ the 2-period problem
   * $\Rightarrow$ the principle of optimality